Landau Damping

Plasma Physics

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Wave propagation in warm plasmas

- Assumptions
  - Small amplitude
  - Uniform plasma
  - No magnetic field
  - Ions are immobile, only $\bar{e}$ move

Wave propagation in warm plasmas

- Looking for electrostatic plasma waves ($\vec{B} = 0$)
- Vlasov equation for an unmagnetized collisionless plasma

$$\frac{\partial f_e}{\partial t} + \vec{v}.\nabla f_e - \frac{e}{m_e}\vec{E}.\nabla_v f_e = 0$$

where

$f_e(\vec{r}, \vec{v}, t)$ - electron distribution function

$\vec{E} = -\nabla\phi,\ \nabla^2\phi = -\frac{\rho}{\epsilon_0} = -\frac{e}{\epsilon_0}\{n - \int f_e \, \mathrm{d}^3\vec{v}\}$

n is the no. density of ions

Vlasov's solution

- Write f_e as equilibrium + perturbation

$$f_e(\vec{r}, \vec{v}, t) = f_0(\vec{v}) + f_1(\vec{r}, \vec{v}, t)$$

- f_0 is the equilibrium distribution and f_1 is the perturbation
- Only for dealing with small amplitude waves
- Note that

$$\int f_0 \, \mathrm{d}^3\vec{v} = n$$

Vlasov's solution

- Using
$$f_e(\vec{r},\vec{v},t) = f_0(\vec{v}) + f_1(\vec{r},\vec{v},t)$$
and
$$\frac{\partial f_e}{\partial t} + \vec{v}.\nabla f_e - \frac{e}{m_e}\vec{E}.\nabla_v f_e = 0$$
- We get
$$\frac{\partial f_1}{\partial t} + \vec{v}.\nabla f_1 - \frac{e}{m_e}\vec{E}.\nabla_v f_0 = 0$$
and
$$\nabla^2\phi = \frac{e}{\epsilon_0}\int f_1 \ \mathrm{d}^3\vec{v}$$

Vlasov's solution

- Assume all perturbed quantities vary as $\exp\{i(\vec{k}.\vec{r} - \omega t)\}$
- Solve for f_1 from

$$\frac{\partial f_1}{\partial t} + \vec{v}.\nabla f_1 - \frac{e}{m_e}\vec{E}.\nabla_v f_0 = 0$$

and substitute in

$$\nabla^2 \phi = \frac{e}{\epsilon_0}\int f_1 \ \mathrm{d}^3\vec{v}$$

- We get (ϕ has to be non-zero)

$$\frac{e^2}{\epsilon_0 m_e k^2}\int \frac{\vec{k}.\nabla_v f_0}{\omega - \vec{k}.\vec{v}}\mathrm{d}^3\vec{v} \ = \ -1$$

Dispersion relation

- We got the dispersion relation

$$\frac{e^2}{\epsilon_0 m_e k^2} \int \frac{\vec{k}.\nabla_v f_0}{\omega - \vec{k}.\vec{v}} \mathrm{d}^3\vec{v} = -1$$

- But there is a singularity at $\omega = \vec{k}.\vec{v}$
- Landau pointed out this problem
- His solution: do not take time dependence as $\exp\{-i\omega t\}$, treat it as an initial value problem:
- f_1 is given at $t = 0$ and we have to find it out at later times.

Landau's solution

Take:

$$f_1(\vec{r},\vec{v},t) = f_1(\vec{v},t)\exp\{i\vec{k}.\vec{r}\}$$

$$\vec{u} = \frac{\vec{k}.\vec{v}}{k}\hat{k}$$

$$\vec{u_p} = \vec{v} - \vec{u}$$

$$F_0(u) = \int f_0(\vec{v})\ \mathrm{d}^3\vec{u_p}$$

$$F_1(u,t) = \int f_1(\vec{v},t)\ \mathrm{d}^3\vec{u_p}$$

Landau's solution

- Start from

$$\frac{\partial f_1}{\partial t} + \vec{v}.\nabla f_1 - \frac{e}{m_e}\vec{E}.\nabla_v f_0 = 0$$

and

$$\nabla^2 \phi = \frac{e}{\epsilon_0} \int f_1 \, \mathrm{d}^3\vec{v}$$

- Use Laplace transforms of F_1 and E:

$$\bar{F}_1(u,p) = \int_0^\infty F_1(u,t) \mathrm{e}^{-pt} \, \mathrm{d}t$$

$$\bar{E}(p) = \int_0^\infty E(t) \mathrm{e}^{-pt} \, \mathrm{d}t$$

Landau's solution

- Some algebra later...

$$\bar{E} = -\frac{e/\epsilon_0}{ik\epsilon(k,p)} \int_{-\infty}^{\infty} \frac{F_1(u, t=0)}{p + iku} \mathrm{d}u$$

$$\epsilon(k,p) = 1 \; + \; \frac{e^2}{\epsilon_0 m_e k} \int_{-\infty}^{\infty} \frac{\partial F_0/\partial u}{ip - ku} \mathrm{d}u$$

and

$$\bar{F_1} = \frac{e}{m_e} \bar{E} \frac{\partial F_0/\partial u}{p + iku} + \frac{F_1(u, t=0)}{p + iku}$$

Landau's solution

- $\epsilon(k,p)$ is known as the *plasma dielectric function*
- Note that the integrals are well defined
- Now we have to invert these Laplace transforms:

$$a(u,t) = \frac{1}{2\pi i}\int_C \bar{a}(u,p)\mathrm{e}^{pt}\mathrm{d}p$$

where C is the *Bromwich contour*, which is the contour running parallel to the imaginary axis, and lying to the right of all singularities of $\bar{F}_1$ in the complex-p plane.

Landau's solution

- Deformation of contours possible when $F_0(u)$ and $F_1(u, t)$ vary smoothly with u.

- $$E = \frac{1}{2\pi i}\int_C \Big[\ \frac{-e/\epsilon_0}{ik\epsilon(k,p)}\int_{-\infty}^{\infty}\frac{F_1(u,t=0)}{p+iku}\mathrm{d}u\Big]\ \ \mathrm{e}^{pt}\mathrm{d}p$$

  E is dominated by the zeros of $\epsilon(k, p)$.

- $$F_1 = \frac{1}{2\pi i}\int_C \Big[\ \frac{e}{m_e}\bar{E}\frac{\partial F_0/\partial u}{p+iku} + \frac{F_1(u,t=0)}{p+iku}\Big]\ \ \mathrm{e}^{pt}\mathrm{d}p$$

  Same with F_1

Landau's solution

- Behaviour depends almost exclusively on the zeros of $\epsilon(k,p)$

$$\epsilon(k,p) = 1 \; + \; \frac{e^2}{\epsilon_0 m_e k} \int_{-\infty}^{\infty} \frac{\partial F_0/\partial u}{ip - ku} \mathrm{d}u \; = \; 0$$

- If $p = -i\omega$, this equation is the same as what we got initially

$$\frac{e^2}{\epsilon_0 m_e k^2} \int \frac{\vec{k}.\nabla_v f_0}{\omega - \vec{k}.\vec{v}} \mathrm{d}^3\vec{v} \; = \; -1$$

Dispersion relation

- Use Maxwellian distribution for F_0

$$F_0(u) = \frac{n}{\sqrt{2\pi T_e/m_e}} \exp\{\frac{-m_e u^2}{2T_e}\}$$

- Taylor-expanding denominator

$$\frac{1}{\omega - ku} = \frac{1}{\omega}(1 + \frac{ku}{\omega} + \frac{k^2 u^2}{\omega^2} + \ldots)$$

- Treat imaginary part of frequency as a perturbation

$$\omega \approx \Re\{\omega\} = \omega_0, \quad \omega = \omega_0 + \delta\omega$$

Dispersion relation

After some complex analysis and algebra, we finally get

$$\omega^2 \approx \omega_p^2(1 + 3k^2\lambda_D^2)$$

and

$$\begin{aligned}\delta\omega &\approx \frac{i\pi}{2}\frac{e^2\omega_p}{\epsilon_0 m_e k^2}\left(\frac{\partial F_0}{\partial u}\right)_{u=\omega/k} \\ &\approx \frac{-i}{2}\sqrt{\frac{\pi}{2}}\frac{\omega_p}{(k\lambda_D)^3}\exp\left\{\frac{-1}{2(k\lambda_D)^2} - \frac{3}{2}\right\}\end{aligned}$$

where λ_D is the Debye length and ω_p is the plasma frequency

$$\lambda_D = \sqrt{\frac{T_e}{m_e\omega_p^2}} \quad \text{and} \quad \omega_p = \sqrt{\frac{ne^2}{\epsilon_0 m_e}}$$

Landau damping

- Damping happens due to the imaginary term

$$\delta\omega \approx \frac{-i}{2}\sqrt{\frac{\pi}{2}}\frac{\omega_p}{(k\lambda_D)^3}\exp\{\frac{-1}{2(k\lambda_D)^2} - \frac{3}{2}\}$$

  This is the *Landau damping*.

- Large for shorter wavelengths, negligible for longer wavelengths.

Landau damping: Verification

- Equation of motion of a charged particle:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \frac{e}{m} E_0 \exp\{i(kx - \omega t)\}$$

- Taking $x = x_0 + u_0 t$ in the electric field, we get

$$u - u_0 = \frac{\mathrm{e}^{ikx_0} e E_0}{i(ku_0 - \omega)m}[\exp\{i(ku_0 - \omega)t\} - 1]$$

Landau damping: Verification

- Particles far from $u_0 = \omega/k$ will simply oscillate.
- If $u_0 \approx \omega/k$, then

$$u - u_0 = \frac{eE_0}{m} t \mathrm{e}^{ikx_0}$$

These are *resonant* particles, some of them gain energy from the wave, and some lose energy to the wave.

Landau damping: Verification

- For a particle with u_0 slightly greater than ω/k

  — loses energy $\implies$ comes close to ω/k (more interaction)

  — gains energy $\implies$ goes away from ω/k

- For a particle with u_0 slightly lesser than ω/k

  — loses energy $\implies$ it goes away from ω/k

  — gains energy $\implies$ it comes close to ω/k (more interaction)

- On average

  — particles with $u_0 \gtrapprox \omega/k$ lose energy

  — particles with $u_0 \lessapprox \omega/k$ gain energy

Landau damping

- Intuition: Surfers on a wave
- If more particles have $u_0 \lessapprox \omega/k$ than particles with $u_0 \gtrapprox \omega/k$:
  — $\implies$ net transfer of energy from $\vec{E}$ to the particles
  — i.e., the electric field is damped.
- Similar effect in galactic dynamics:
  — Electrons $\leftrightarrow$ stars, electric field $\leftrightarrow$ gravity
  — Vlasov eqn. is called *Collisionless Boltzmann eqn.*

References

- Plasma Physics web notes, R. Fitzpatrick, U. Texas Austin

  http://farside.ph.utexas.edu/teaching/plasma/Plasmahtml/

- Wikipedia

  https://en.wikipedia.org/wiki/Landau_damping