Fluid dynamics and Plasma physics AA 363 2:0 credits

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1. **Stream function for a line vortex in an incompessible fluid**

Consider a two dimensional infinite XY-plane embedded with a line vortex. The vorticity, $\omega\hat{z}$, of that line vortex centerred at the origin is given by,

$$\omega(\mathbf{r}) = \frac{\Gamma}{2\pi\rho}\delta(\rho)$$

Where, $\rho = \sqrt{x^2 + y^2}$.

(a) Calculate the streamline function, Ψ, for this flow. Plot the azimuthal velocity as a function of ρ using your favorite plotting software.

(b) Now replace the δ functions with Gaussian blobs of width ϵ, successively of second, fourth and sixth order in ϵ as,

$$\omega(\rho) = \xi(\rho) = \frac{\Gamma}{\pi\epsilon^2}\exp(-\rho^2/\epsilon^2)$$

$$\omega(\rho) = \xi_4(\rho) = \frac{\Gamma}{\pi\epsilon^2}(4 - \frac{\rho^2}{\epsilon^2})\exp(-\rho^2/\epsilon^2)$$

$$\omega(\rho) = \xi_6(\rho) = \frac{\Gamma}{\pi\epsilon^2}(6 - 6\frac{\rho^2}{\epsilon^2} + \frac{\rho^4}{\epsilon^4})\exp(-\rho^2/\epsilon^2)$$

and overplot the corresponding azimuthal velocities as a function of ρ (using different color or linestyles or widths) over the velocities from the Green's function plotted earlier. Vary $0 < \epsilon \leq 1$ to show how well the velocities around the line vortex can be reproduced.

(c) Instead of centering the blob vortex at the origin, we can center it at any point inside the domain, (x_0, y_0), then the cylindrical radius is given by $\rho = \sqrt{(x - x_0)^2 + (y - y_0)^2}$. Now vary $(x_0, y_0) = (\pm a, \pm b)$ as well as sign of Γ to produce two line vortices. Trace the flow field. Do you think this is a steady flow field? How will vortices of like and opposite signs behave, respectively? Can you relate it with anything you have studied in electrodynamics?

Hint: Spike on z-axis in cylindrical coordinates, the Dirac's δ-function is represented as,

$$\delta(\mathbf{r} - \mathbf{r}') = \frac{1}{2\pi\rho}\delta(\rho)$$

and the solution of the Poisson equation, $\nabla^2 G = \delta(\mathbf{r} - \mathbf{r}')$, is given by the Green's function, $G(\mathbf{r} - \mathbf{r}')$. The velocity, $\mathbf{v}$, is then given by $\mathbf{v} = \nabla \times (\Psi\hat{z})$

2. **Dual stream functions in three dimensions**

   For an incompressible flow in 3D, we can always write a velocity potential, $\mathbf{\Phi}$, such that,

   $$\mathbf{u} = \nabla \times \mathbf{\Phi}$$

   $\mathbf{\Phi}$ can be further expressed as $\psi \nabla \eta$.

   (a) Using vector calculus, show that the velocity, $\mathbf{u}$, is given by

   $$\mathbf{u} = \nabla \psi \times \nabla \eta$$

   (b) Consider the following expressions,

   $$\psi = \frac{(\rho - \rho_0)^2}{2} + \frac{(z - z_0)^2}{2}$$

   $$\eta = a \tan^{-1}\left(\frac{\rho - \rho_0}{z - z_0}\right) - b\theta$$

   with, $\rho = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(\frac{y}{x})$. Choose, $\rho_0 = 20, z_0 = 5, a = 20, b = 2$. Draw the surfaces of $\psi = C_1$ and $\eta = C_2$, where C_1 and C_2 are suitably chosen constants.

   (c) Calculate the 3D velocity field and plot a few streamlines of the flow in 3-dimensions in the same figure as (b). Verify if the intersection of the two surfaces corresponds to any streamline.

3. **Same Fluid equations, different variables**

   The conservative form of the mass conservation equation or the continuity equation is given as

   $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = \mathbf{0}$$

   (a) Write this equation in terms of $\ln \rho$, c_s and $\mathbf{u}$ for an ideal gas, where c_s is the speed of sound. Is the new equation in conservative form? Think of a reason or situation when evolving $\ln \rho$ in time is more beneficial than evolving just ρ.

   (b) Similarly, the Euler's velocity equation is given by

   $$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\nabla p}{\rho} + \mathbf{f}$$

where, $\gamma p = \rho c_s^2$ is the ideal gas equation with $\gamma = 5/3$. and $\mathbf{f} = -\nabla\phi$, a conservatve force, like gravity. Rewrite this equation in terms of $\ln\rho$, c_s^2, $\mathbf{u}$ and ϕ.

(c) The final equation - the energy equation - can have different forms depending on the system and the problem at hand. One can either write an equation for the total energy, E, or internal energy e (per unit mass), pressure, p, temperature, T, or specific entropy s. Consider an ideal gas system with adiabatic index, γ and write an evolution equation for the for specific entropy, s stating from the equation specific internal energy, e or internal energy per unit volume, ρe.

4. **Hydrostatic balance in an ideal vs ionized gas**

Consider partially ionized gas consisting of H, H^+, and He in the solar atmosphere. We will work in Cartesian coordinates where z denotes the height above the photosphere. For the time being let us neglect the contribution of elements heavier than He to the gas pressure. The effective atomic weight in amu, μ is given by,

$$\mu^{-1} = (1 + y_H + x_{He})/(1 + 4x_{He})$$

Here, y_H is the Hydrogen ionisation fraction, $x_{He} = 0.1$ is the Helium fraction and then the pressure will be, according to the ideal gas law,

$$p = \frac{R_{\text{gas}}\rho(z)T(z)}{\mu}$$

Also denote,

$$T_{\text{ion}} = \chi_B/k_B$$

where, χ_B =13.6 eV (convert to Joules) and $k_B = 1.3806505 \times 10^{-23}$ J K^{-1} is the Boltzman constant.

1 eV= $1.602176462 \times 10^{-19}$ J, $m_u = 1.6053886 \times 10^{-27}$ kg, $R_{\text{gas}} = k_B/m_u$, electron mass $m_e = 9.11 \times 10^{-31}$ kg

Let us define,

$$\xi = \frac{\rho_e}{\rho(z)}\left(\frac{T_{\text{ion}}}{T(z)}\right)^{-3/2}\exp(-T_{\text{ion}}/T(z))$$

where

$$\rho_e = m_u(1 + 4x_{He})\left(\frac{1}{2\pi}\frac{m_e\chi_B}{\hbar^2}\right)^{3/2}$$

Then, the ionization fraction of Hydrogen from Saha's ionization formula is,

$$y_H = \frac{2\sqrt{\xi}}{\sqrt{\xi} + \sqrt{4+\xi}}$$

(a) Given the solar temperature profile (in log scale), $\ln T(z)$ as a function of z (Mm) (as an ascii data file), calculate the y_H and the density profile numerically using the equation of hydrostatic balance. *Hint:* Use an iterative method for example. You may use the density at the bottom boundary, $\rho_0 = 2 \times 10^{-2}$ kg m^{-3}.

(b) For the same temperature, $T(z)$, put $y_H = 1$ and re-calculate the density $\rho(z)$, Plot both density curves in a single plot. Can you see the error introduced by using full ionization? Here, also use the same ρ_0 as in (a) for calibration.