One-Fluid Model for Plasma 2: The Conclusion

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Outline

- One-fluid model
- Generalized Ohm’s law
- Hall effect
- Basic idea of MHD
For length-scales larger than Debye length and timescales larger than the inverse of plasma frequency, charge separation can be neglected and a fully ionized plasma can be regarded as a single fluid.

Collisions cannot be neglected in these large scales.
Consider an electron fluid. Due to collisions, the momentum transfer is dominant between electrons and ions, than among the electrons.

- This loss of momentum for an electron due to an encounter with an ion is \( m_e n (v_e - v_i) \)
- Equation of motion for an electron fluid is given by,

\[
  m_e n \frac{\partial v_e}{\partial t} = -\nabla p_e - n e(\frac{v_e}{c} \times B) - m_e n v_c (v_e - v_i) \tag{1}
\]

- Current density relation

\[
  j = n e(v_i - v_e) \tag{2}
\]
Substituting Eq. 1 in Eq. 2,

\[ m_e n \frac{\partial v_e}{\partial t} = -\nabla p_e - ne(E + \frac{v_e}{c} \times B) + ne\eta j, \] (3)

where \( \eta \) is given by

\[ \eta = \frac{m_e \nu_c}{ne^2} \] (4)

the collision term in the equation of the ion fluid should be equal to that for the electron fluid in (3) with an opposite sign, i.e

\[ m_i n \frac{\partial v_i}{\partial t} = -\nabla p_i + ne(E + \frac{v_i}{c} \times B) - ne\eta j, \] (5)
On adding the equations 3, 5 we get

\[ n \frac{\partial}{\partial t}(m_i v_i + m_e v_e) = \frac{ne(v_i - v_e)}{c} \times B - \nabla(p_i + p_e) \quad (6) \]

The density \( \rho \) and the fluid velocity \( v \) in the single-fluid model are given by

\[ \rho = n(m_i + m_e) \quad (7) \]
\[ v = \frac{(m_i v_i + m_e v_e)}{m_i + m_e} \quad (8) \]

Using the equations 6, 7 and 8 we get,

\[ \rho \frac{\partial v}{\partial t} = \frac{1}{c} j \times B - \nabla P \quad (9) \]
To obtain the relation we have considered,

$$p = p_i + p_e$$  \hspace{1cm} (10)

Now multiply Eq. 3 by $m_i$ and subtract it from Eq. 5 multiplied by $m_e$. This gives,

$$m_i m_e n \frac{\partial(v_i - v_e)}{\partial t} = n e (m_i + m_e) E + \frac{n e}{c} (m_e v_i + m_i v_e) \times B$$

$$-m_e \nabla P_i - m_i \nabla P_e - (m_i - m_e) n e \eta j$$  \hspace{1cm} (11)
Using the Equations (2),(7),(8) we can obtain the relation,

\[ m_e v_i + m_i v_e = m_i v_i + m_e v_e + (m_e - m_i)(v_i - v_e) \]

\[ = \frac{\rho}{n} v + \frac{m_e - m_i}{ne} j \]  \hspace{1cm} (12)

Using this with the Eq 11 we get, The generalised Ohms law

\[ E + \frac{v}{c} \times B - \eta j = \frac{1}{\epsilon \rho} \left[ \frac{m_i m_e n}{e} \frac{\partial}{\partial t} \left( \frac{j}{n} \right) + (m_e - m_i) \frac{j}{c} \times B + m_e \nabla P_i - m_i \nabla P_e \right] \]  \hspace{1cm} (13)
The equation then simplifies into

$$E + \frac{v}{c} \times B - \eta j = \frac{1}{ne} \left[ \frac{j}{c} \times B - \nabla P_e \right]$$  \hspace{1cm} (14)

Term involving \(j \times B\) corresponds to the Hall effect

The equation can be further simplified into Ohm’s law

$$j = \sigma (E + \frac{v}{c} \times B)$$ \hspace{1cm} (15)

where electrical conductivity,

$$\sigma = \eta^{-1}$$ \hspace{1cm} (16)
The single-fluid model of the plasma is known as MHD.

\[ j = \sigma (E + \frac{v}{c} \times B) \]  \hspace{1cm} (17)

\[ \rho \frac{\partial v}{\partial t} = \frac{1}{c} j \times B - \nabla P \]  \hspace{1cm} (18)

These equations used in further development of MHD in the later chapter

The equations of continuity for the electron and ion fluids give the one-fluid equation of continuity

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0 \]  \hspace{1cm} (19)