Fluid dynamics and Plasma physics AA 363 2:0 credits
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Homework Problem collection – DEADLINE 15 Oct, 2022

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FIG. 1. Vehicle movement along a winding road captured by long exposure times

1. **Moving light photography [5 marks]** Assume the speeds of the cars be 40 km/hr. The size of the car is typically 4 m and headlight size would be 15 cm. A car passes a given point every 9 s. And the length of the winding road covered in the photographer's frame is 2 km. Calculate the exposure times needed to obtain the lighted track lines like that shown in Fig. 1. The photographers claim the exposure times to be 20 s. Do you also obtain a similar number? How does the exposure time depend on the density of the traffic?

   TIPS: Remember to use a low ISO to reduce noise and an adequate aperture to record some of the comparatively dimmer background.

2. **Maxwell-Boltzman distribution [6 marks]** Using the assumption of molecular chaos and vanishing collision integral in the Boltzman equation, show that the relevant zeroeth order distribution function is given by

$$f_0(\vec{p}) = n\left(\frac{3}{4\pi m_p \epsilon}\right)^{3/2} \exp[-3(\vec{p}-\vec{p}_0)^2/(4m_p\epsilon)]$$

   Here, m_p is the particle mass, ϵ is the expectation value of kinetic energy per particle, n is the expectation value of number density of particles.

   Next, show that $f_0(\vec{p})$ does not satisfy the streaming part (LHS) of the Boltzman equation.

3. **Pressure and transport properties from Boltzman equation [10 marks]**

   In order to satisfy the Boltzman equation, we add a first order correction to the MB distribution, namely $f = f_0 + g$, such that $g << f_0$. And approximated by,

$$g = -\tau\left[\frac{\partial f_0}{\partial t} - \{\mathcal{H}, f_0\}\right].$$

   where, τ is the relaxation time. After rearrangement the above equation is given by, [see §5.5 Kerson Huang]

$$g = -\tau\left[\frac{1}{\theta}\frac{\partial\theta}{\partial x_i}U_i\left(\frac{m_p}{2\theta}U^2 - \frac{5}{2}\right) + \frac{1}{\theta}\Lambda_{ij}(U_iU_j - \frac{1}{3}\delta_{ij}U^2)\right] f_0$$

   with $\theta = \frac{2}{3}\epsilon$, $U_i = v_i - u_i$ and $U = |\vec{v} - \vec{u}|$. $\Lambda_{ij} = \frac{m_p}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$. Note that we use the same symbols as in class, i.e., v_i denotes particle velocity and $u_i(\vec{x}, t)$ denotes the expectation value of velocity or the fluid velocity at $(\vec{x}, t)$.

   The pressure of the collection of gas particles then is given by,

$$P_{ij} = \rho\langle(v_i - u_i)(v_j - u_j)\rangle.$$

(a) Use the definition of expectation values to expand this and show that

$$P_{ij} = \delta_{ij}\frac{\rho\theta}{m_p} + P'_{ij}$$

Here, P'_{ij} is the correction to the isotropic pressure.

(b) Use the expression for first order correction, g, above to show that

$$P'_{ij} = -\frac{\tau\rho m_p^3}{\theta n}\Lambda_{kl}\int d^3U U_i U_j (U_k U_l - \frac{1}{3}\delta_{kl}U^2) f_0$$

(c) Using the zero trace property of P'_{ij}, deduce that a simple expression is

$$P'_{ij} = -\frac{2\mu}{m_p}\left(\Lambda_{ij} - \frac{m_p}{3}\delta_{ij}\nabla\cdot\vec{u}\right).$$

where, $\mu = \tau n\theta$, is known as the coefficient of viscosity.

4. **Mass conservation equation from Boltzmann transport equation [6 marks]**

   Consider the Boltzmann transport equation for a collection of classical particles of mass m_p, with velocity ,$\vec{v}$, and acceleration, $\vec{a}$ at any point $\vec{x}$.

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\nabla f + \vec{a}\cdot\nabla_u f = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}}$$

   Now the gradient operator, $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$, Likewise, $\nabla_u = (\frac{\partial}{\partial v_x}, \frac{\partial}{\partial v_y}, \frac{\partial}{\partial v_z})$.

   The density in the fluid description is then defines as,

$$\rho(\vec{x}) = m_p\int f d^3\vec{v}$$

   Now using the above concepts, prove the equation of mass conservation for fluids namely,

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0.$$

   TIP: In these classical hard sphere collision where mass of the colliding particles remains unchanged, integrals like, $\int\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} d^3\vec{v} = 0$.

5. **Advective derivative [3 marks]**

   There are two observers, one at a weather station at a point x and another passing by in a hot air balloon, horizontally. The observer at the station notices that the temperature is falling at rate of 1 C day^{-1}, while the balloon bound weather man doesn't observe any change at all. If the balloon is moving east at a constant speed of 10 ms^{-1}, what can you conclude about the spatial distribution of background temperature?

6. **Momentum equation [5 marks]** Derive the velocity equation for compressible inviscid fluid using Lagrangian approach, i.e.,

$$\rho\frac{D\vec{v}}{dt} = -\nabla p + \vec{F},$$

   where, $\vec{F}$ is a body force acting on any volume of the fluid.

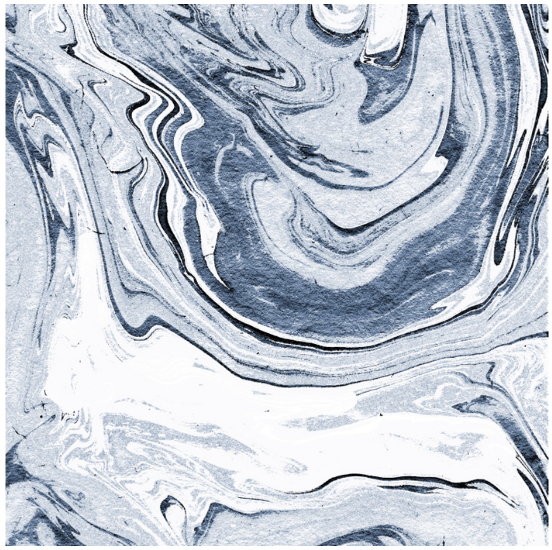

FIG. 2. Ink drift painting - The Japanese art of *Suminagashi*

7. **Equation for the conservation of energy [6 marks]** The primitive equation for the time-evolution of specific entropy is given by,

$$T\rho\frac{Ds}{Dt} = \dot{\mathcal{H}} - \dot{\mathcal{C}} + \nabla \cdot (\kappa \nabla T)$$

Let, the specific energy denoted, e. Combine this first law of thermodynamics and with the equation in Q. (6) to show that,

$$\frac{\partial}{\partial t}\left[\rho(e + \frac{v^2}{2})\right] + \nabla \cdot \left[\rho\vec{v}(e + \frac{v^2}{2} + p/\rho)\right] = T\rho\frac{Ds}{Dt} + \vec{F}\cdot\vec{v}$$

8. **Flow Visualization [3 marks]** Take a tray full of water and stir the surface, gently so that the motion is smooth with some eddies. On the distorted surface of this water quietly drop some ink. After sometime place a sheet of water on the surface and the impression transferred on the paper will look similar to the picture shown in Fig. 2. This picture is a snapshot at a particular time and consists of lots of curves. Are these streamlines, particle lines or streak lines, or any other kind of lines?

9. **Dual stream functions in three dimensions**[6 marks]

For an incompressible flow in 3D, we can always write a velocity potential, $\mathbf{\Phi}$, such that,

$$\mathbf{u} = \nabla \times \mathbf{\Phi}$$

$\mathbf{\Phi}$ can be further expressed as $\psi\nabla\eta$.

(a) Using vector calculus, show that the velocity, $\mathbf{u}$, is given by

$$\mathbf{u} = \nabla\psi \times \nabla\eta$$

(b) Consider the following expressions,

$$\psi = \frac{(\rho - \rho_0)^2}{2} + \frac{(z - z_0)^2}{2}$$

$$\eta = a\tan^{-1}\left(\frac{\rho - \rho_0}{z - z_0}\right) - b\theta$$

with, $\rho = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(\frac{y}{x})$. Choose, $\rho_0 = 20, z_0 = 5, a = 20, b = 2$. ψ and η functions will be constant on surfaces as this is 3D. Draw the surfaces of $\psi = \eta = C_1$, where C_1 is a suitably chosen constant.

(c) Calculate the 3D velocity field and plot a few streamlines of the flow in 3-dimensions in the same figure as (b). Show that the intersection of the two surfaces ($\psi = \eta = C_1$, say) corresponds to a streamline.

10. **Line vortex pairs**[5 marks]

    Show that a pair of line vortices with strength $\pm\Gamma/2$ respectively and centers a distance d apart, will move in a direction perpendicular to the line joining their centers at a velocity $\Gamma/4\pi d$. Also plot the instantaneous streamlines in a frame moving with the vortices.

11. **Helmholtz first vorticity theorem** [5 marks]

    Show that a fluid element lying on a vortex line continues to lie on a vortex line as the invisid and incompressible flow evolves in time.

12. **Kinetic helicity as an invariant of motion**[5 marks]

    Show that when the fluid is inviscid and of constant density, the kinetic helicity, $\mathcal{H}_K$ is an invariant of motion, i.e., $d\mathcal{H}_K/dt = 0$. Where,

    $$\mathcal{H}_K = \int_V \vec{v}\cdot\vec{\omega}dV$$

    Assume that there are no fluxes out of the fixed volume, V, and the $|\vec{v}| \to 0$ like $O(1/r^3)$ for $r \to \infty$ (Biot-Savart Law for vorticity). Because of this the surface integrals vanish.

    [Hint:] For two arbitrary vectors $\vec{A}$ and $\vec{B}$ decaying to zero like $\vec{v}$, we can use the equality:

    $$\int \vec{A}\cdot(\nabla\times\vec{B})dV = \int(\nabla\times\vec{A})\cdot\vec{B}dV$$

    Now, set $\vec{A} = \vec{v}$ and $\vec{B} = \partial_t\vec{v}$.

13. **Rossby number in bath-tub draining flows [3 marks]**

    Calculate the Rossby number of the flow out of a tank drain. Assume the tank is 2 m in height with a hole 1 cm in diameter at the bottom and is located at a latitude of 45^o N.

14. **Tornados and circulation [6 marks]** A tornado is a vortex with a core diameter, $2a = 50$ m. The vorticity $\omega = \omega_0/2$, a constant for cylindrical radius, $\rho < a$ (see Fig. 3).

    (a) Derive an expression (in cylindrical geometry) for the azimuthal velocity, u_θ for any $0 < \rho < \infty$.

    (b) The gauge pressure at a radius of 25 m from center is, $P_a = P_{\text{atm}} - 2500\text{Nm}^{-2}$, $P_{\text{atm}} = 10^5$ N m^{-2}. Using the result in (a) show that the circulation, Γ around any circuit surrounding the core i.e., for $\rho > a$ is given by, 10140 m^{2}s^{-1}. Using this calculate the value of ω_0 and therefore the Rossby number inside the tornado. (*Hint*: Use Bernoulli's theorem for incompressible steady flows between infinity and edge of the tornado core.) (c) If the ground speed of such a tornado is 25 ms^{-1}, calculate the time required for the pressure to drop at any location on its trajectory from $P_1 = P_{\text{atm}} - 500$ Nm^{-2} to $P_2 = P_{\text{atm}} - 2500$ Nm^{-2}.

15. **Prandtl-Batchelor theorem [5 marks]**

    Consider the vorticity equation and its consequences for a steady 2-D incompressible flow (say in xy-plane). Essentially we have for a flow with velocity, **u** and corresponding vorticity (in z-direction), ω,

    $$(\mathbf{u}.\nabla)\omega = \mathbf{0}$$

    This means that ω is constant along a streamline defined by the streamfunction, ψ and hence $\omega \equiv (0, 0, \omega(\psi))$, i.e., a unique function of the streamfunction. For open field lines traced till infinity this means that if $\omega = 0$ at infinity then it will remain zero on that streamline everywhere. Now, what about streamlines that are closed?

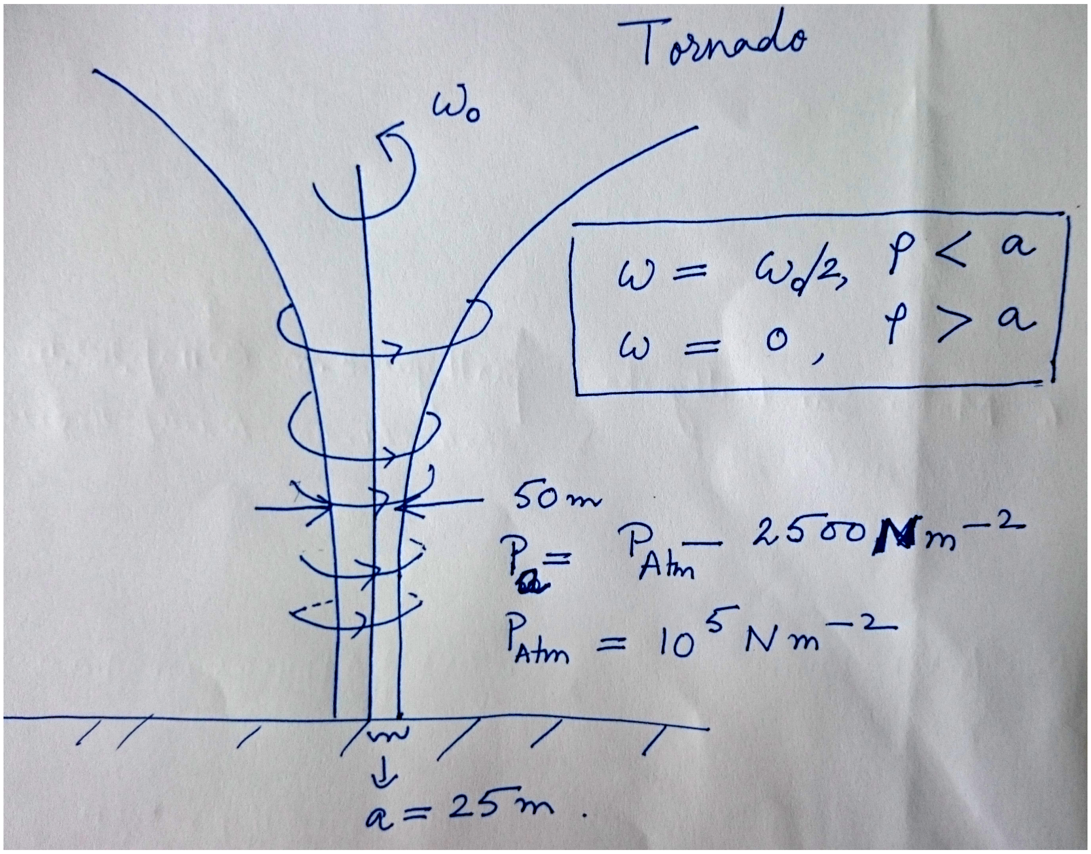

FIG. 3. Illustration for Q. 12

For truly invisid flows we can't say anything, but for flows with finite viscosity, ν, however small, *the Prandtl-Batchelor theorem predicts the distribution of* ω *inside the closed streamline region.* To find what the theorem says, start from the Navier-Stokes equation for steady flows, namely,

$$(\mathbf{u}.\nabla)\mathbf{u} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{u} \tag{1}$$

and rewrite Eq. (1) using suitable vector calculus identities and then integrate both sides around a closed streamline, C, to show that

$$\nu \oint_C \nabla \times \omega . d\mathbf{l} = \mathbf{0} \tag{2}$$

Next, verify that

$$\nabla \times (0, 0, \omega(\psi)) = \frac{\partial \omega}{\partial \psi}(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0)$$

and combining with Eq. (2), for finite ν, show that

$$\frac{\partial \omega}{\partial \psi} \oint_C \mathbf{u}.d\mathbf{l} = 0 \tag{3}$$

What distribution does Eq. (3) imply for closed streamline regions?